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Identification of Modal Parameters from Nonstationary Ambient Vibration Data Using Correlation Technique

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Identification of modal parameters is considered from response data of structural systems under nonstationary ambient vibration. It is shown theoretically that by assuming the ambient excitation to be nonstationary white noise in the form of a product model, the nonstationary response signals can be converted into free-vibration data via the correlation technique. The practical problem of insufficient data samples available for evaluating nonstationary correlation can be approximately resolved by first extracting the amplitude-modulating function from the response and then transforming the nonstationary responses into stationary ones. Modal-parameter identification can then be performed using the Ibrahim time-domain technique, which is effective at identifying closely spaced modes. The theory proposed can be further extended by using the filtering concept to cover the case of nonstationary color excitations. Numerical simulations confirm the validity of the proposed method for identification of modal parameters from nonstationary ambient response data.

Nomenclature

$\boldsymbol{A}$	=	diagonal matrices with elements
a_r	=	diagonal element of $\boldsymbol{A}$
$\boldsymbol{B}$	=	diagonal matrices with elements
b_r	=	diagonal element of $\boldsymbol{B}$
$\boldsymbol{C}$	=	system damping matrix
$\boldsymbol{f}(t)$	=	excitation vector
$\boldsymbol{K}$	=	system stiffness matrix
$\boldsymbol{l}$	=	vector for which the elements are the influence factors for each degree of freedom
$\boldsymbol{M}$	=	system mass matrix
$\boldsymbol{q}(t)$	=	vector of modal coordinate
$R_{ijk}(t,\tau)$	=	cross-correlation function between two nonstationary response signals $x_{ik}(t)$ and $x_{jk}(t)$
$\boldsymbol{S}_{(n\times n)}$	=	system matrix
T	=	short time interval
$\hat{u}_i^2(t)$	=	temporal mean-square function
$\boldsymbol{v}(t)$	=	stationary displacement response
$w_k(t)$	=	stationary white noise
$\boldsymbol{X}_{(n\times q)}$	=	$n \times q$ matrix of measured response
$\boldsymbol{x}(t)$	=	displacement vector
$x_{ik}(t)$	=	response at the ith degree of freedom due to the input at the kth degree of freedom
$\boldsymbol{Y}_{(n\times q)}$	=	$n \times q$ matrix of time-delayed response
α_k	=	white-noise spectral density constant associated with the kth input
$\Gamma_k(t)$	=	deterministic amplitude-modulating function associated with the kth input
$\delta(t)$	=	Dirac delta function
$\boldsymbol{\Psi}$	=	complex modal matrix
ψ_{ir}	=	ith component of the rth mode shape
$\boldsymbol{\Psi_r}$	=	vector of the rth mode shape

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Introduction

EXPERIMENTAL identification of modal parameters of a structure is usually carried out by measuring both its input and corresponding output. Some modal testing techniques use free or impulse responses of structures so that the input excitation need not be measured. However, there are situations in which controlled excitation or initial excitation cannot be employed, such as the case of in-operation testing or in-flight measurement. Consequently, it is desirable to develop techniques for modal-parameter identification without the need of input measurement.

Modal-parameter identification from ambient vibration data has gained considerable attention in recent years [1,2]. The use of ambient vibration survey for determining dynamic characteristics of engineering structures is a valuable tool for practical structural health monitoring [3,4]. A variety of methods have been developed for extracting modal parameters from structures undergoing ambient vibration. Akaike [5] was the first to use the autoregressive moving average (ARMA) model to analyze systems with ambient vibration. Cremona and Brandon [6] presented a modal-identification algorithm based on the ARMA model for the case of unmeasured input. In general, the methods based on the ARMA model can identify modal frequencies and damping ratios effectively, but the identification of mode shapes may become a problem, especially when the number of measured degrees of freedom is large [7]. Ibrahim [8] applied the random decrement technique coupled with a time-domain parameter identification method (ITD) [9] to process ambient vibration data. Although the random decrement technique serves as an alternative method for estimating the autocorrelation and cross-correlation functions [10], it is based on an intuitive theory and does not yet have sound mathematical basis for general cases [11]. James et al. [12] developed the so-called natural excitation technique (NExT) using the cross-correlation technique coupled with time-domain parameter extraction. It was shown that the cross-correlation between two response signals of a linear system with classical normal modes and subject to white-noise (stationary) inputs is of the same form as free-vibration decay or impulse response. When coupled with a time-domain modal-extraction scheme, this concept becomes a very powerful tool for the analysis of structures under ambient vibration. NExT has been applied to modal identification of many engineering structures, such as wind turbines [13] and a rocket during launch [14].

In the previous studies of modal-parameter identification from ambient vibration data, the assumption usually made is that the input excitation is a broadband stochastic process modeled by *stationary* white or filtered white noise. In the present paper, a theoretical justification of the cross-correlation technique is presented for general linear systems excited by *nonstationary* white noise. It is shown that the nonstationary correlation functions evaluated at an arbitrary fixed time instant of structural response are of the same form as free-vibration decay of the structure with certain initial conditions. Therefore, by treating the sample correlations of measured response corresponding to some fixed time instant as output from free-vibration decay, a time-domain modal-identification method, such as the ITD method [9], can then be employed to extract modal parameters, including modal frequencies, damping ratios, and mode shapes of the structure with complex modes.

Theoretical Development of Correlation Technique

The standard matrix equations of motion of a discrete linear system can be expressed in state space as

$$\begin{bmatrix} \boldsymbol{C} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{O} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{x}}(t) \\ \ddot{\boldsymbol{x}}(t) \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{M} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{0} \end{Bmatrix} \tag{1}$$

or

$$\boldsymbol{D}\,\dot{\boldsymbol{y}}(t) + \boldsymbol{E}\boldsymbol{y}(t) = \boldsymbol{p}(t) \tag{2}$$

where

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{O} \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{M} \end{bmatrix}, \quad \boldsymbol{y}(t) = \begin{Bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{Bmatrix}, \quad \boldsymbol{p}(t) = \begin{Bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{0} \end{Bmatrix} \tag{3}$$

In Eq. (3), $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ are, respectively, the system mass, damping, and stiffness matrices, and $\boldsymbol{f(t)}$ and $\boldsymbol{x(t)}$ are the excitation and displacement vectors, respectively. Introduce the transformation as follows:

$$\boldsymbol{y}(t) = \boldsymbol{\Psi}\boldsymbol{q}(t) = \sum_{r=1}^{2m} \boldsymbol{\psi}_r q_r(t) \tag{4}$$

where $\boldsymbol{\Psi}$ denotes the complex modal matrix, $\boldsymbol{q}(t)$ is the vector of modal coordinates, and $\boldsymbol{\psi}_r$ denotes the vector of the rth mode shape. Premultiply Eq. (2) by $\boldsymbol{\Psi}^T$, and Eq. (2) can be transformed to modal coordinates from physical coordinates as follows:

$$\dot{q}_r(t) - \lambda_r q_r(t) = \frac{1}{a_r}\boldsymbol{\psi}_r^T \boldsymbol{p}(t), \qquad r = 1 \sim 2m \tag{5}$$

where we used the orthogonality of mode shapes:

$$\boldsymbol{\Psi}^T\boldsymbol{D}\boldsymbol{\Psi} = \boldsymbol{A}, \qquad \boldsymbol{\Psi}^T\boldsymbol{E}\boldsymbol{\Psi} = \boldsymbol{B} \tag{6}$$

so that $\boldsymbol{A}$, $\boldsymbol{B}$ are diagonal matrices with elements a_r and b_r, $r = 1 \sim 2m$, and

$$\lambda_r = -(b_r/a_r) \tag{7}$$

If we assume that the system is initially at rest, the solution to Eq. (5) can be found as

$$q_r(t) = e^{\lambda_r t}\int_{-\infty}^{t} e^{-\lambda_r \tau}\frac{1}{a_r}\boldsymbol{\psi}_r^T\boldsymbol{p}(\tau)\,\mathrm{d}\tau \tag{8}$$

The following equation can then be derived from Eqs. (4) and (8):

$$\boldsymbol{y}(t) = \sum_{r=1}^{2m}\frac{\boldsymbol{\psi}_r}{a_r}\int_{-\infty}^{t} e^{\lambda_r(t-\tau)}\boldsymbol{\psi}_r^T\boldsymbol{p}(\tau)\,\mathrm{d}\tau \tag{9}$$

It follows from Eq. (9) that the response at the ith degree of freedom (DOF) due to the input at the kth DOF is

$$x_{ik}(t) = \sum_{r=1}^{2m}\frac{\psi_{ir}\psi_{kr}}{a_r}\int_{-\infty}^{t} e^{\lambda_r(t-\tau)} f_k(\tau)\,\mathrm{d}\tau \tag{10}$$

where ψ_{ir} denotes the ith component of the rth mode shape.

Define the cross-correlation function $R_{ijk}(t,\tau)$ between two nonstationary response signals $x_{ik}(t)$ and $x_{jk}(t)$ as

$$R_{ijk}(t,\tau) = E[x_{ik}(t+\tau)x_{jk}(t)] \tag{11}$$

Substituting Eq. (10) into Eq. (11), the following equation can be derived:

$$R_{ijk}(t,\tau) = \sum_{r=1}^{2m}\sum_{s=1}^{2m}\frac{\psi_{ir}\psi_{kr}\psi_{js}\psi_{ks}}{a_r a_s} \cdot \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{\lambda_r(t+\tau-\eta)}e^{\lambda_s(t-\eta)}R_{ffk}(\eta,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\eta \tag{12}$$

Assume that $f_k(t)$ is nonstationary white noise in the form of a *product model*: that is,

$$f_k(t) = \Gamma_k(t)w_k(t) \tag{13}$$

where $\Gamma_k(t)$ is a *deterministic* amplitude-modulating function used to describe the change of amplitude with time, and $w_k(t)$ is stationary white noise. The autocorrelation function of excitation force $f_k(t)$ can then be expressed as

$$R_{ffk}(\eta,\sigma) = \Gamma_k(\eta)\Gamma_k(\sigma)E[w_k(\eta)w_k(\sigma)] = \Gamma_k(\eta)\Gamma_k(\sigma)\alpha_k\delta(\eta-\sigma)$$

where α_k is a constant containing the idealized white-noise spectral density with constant magnitude, and $\delta(t)$ is the Dirac delta function. Therefore, Eq. (12) can be evaluated as

$$R_{ijk}(t,\tau) = \sum_{r=1}^{2m}\sum_{s=1}^{2m}\frac{-\Gamma_k^2(t)\alpha_k\psi_{ir}\psi_{kr}\psi_{js}\psi_{ks}}{a_r a_s(\lambda_r+\lambda_s)}e^{\lambda_r\tau} \tag{14}$$

Using Eq. (14) and summing over all input locations, we obtain

$$R_{ij}(t,\tau) = \sum_{r=1}^{2m}\psi_{ir}\sum_{s=1}^{2m}\sum_{k=1}^{N}\frac{-\Gamma_k^2(t)\alpha_k\psi_{kr}\psi_{js}\psi_{ks}}{a_r a_s(\lambda_r+\lambda_s)}e^{\lambda_r\tau} \tag{15}$$

which can be recast into the form

$$R_{ij}(t,\tau) = \sum_{r=1}^{2m}A_{jr}(t)\psi_{ir}e^{\lambda_r\tau} \tag{16}$$

where $A_{jr}(t)$ can be defined as

$$A_{jr}(t) = \sum_{s=1}^{2m}\sum_{k=1}^{N}\frac{-\Gamma_k^2(t)\alpha_k\psi_{kr}\psi_{js}\psi_{ks}}{a_r a_s(\lambda_r+\lambda_s)} \tag{17}$$

The preceding results show that for any fixed time instant t, $R_{ij}(t,\tau)$ in Eq. (16) is a sum of complex exponential functions, which is of the same form as the free-vibration decay or the impulse response of the original system [9]. Thus, the cross-correlation functions evaluated at a fixed time instant of responses can be used as free-vibration decay or as impulse response in time-domain modal-extraction schemes so that measurement of *nonstationary* white-noise inputs can be avoided. It should be mentioned that the term $A_{jr}(t)\psi_{ir}$ with fixed t in the cross-correlation function of Eq. (16) will be identified as the mode-shape component. To eliminate the $A_{jr}(t)$ term and retain the true mode-shape component ψ_{ir}, all the measured channels are correlated against a common reference channel: say, $x_j(t)$. The identified components then all possess the common $A_{jr}(t)$ component, which can be normalized to obtain the mode shape ψ_{ir}.

Practical Treatment of Nonstationary Data

It has been shown in the previous section that the correlation functions evaluated at a fixed time instant of responses can be used as free-vibration decay or as impulse response in time-domain modal-extraction schemes. However, in engineering practice, very limited data samples are usually available, and so evaluation of the correlation functions could be a significant problem. In the present paper, we try to resolve the problem by first extracting the amplitude-modulating function from the response and then the original nonstationary responses can be transformed into stationary ones.

In the following, we are going to show that if the excitation can be modeled as nonstationary white noise as represented by the product model, then the responses of the system can also be modeled approximately as a product model with the same amplitude-modulating function as that associated with the excitation itself.

We start by considering a discrete linear system subjected to excitation resulted from a single source $w(t)$, which is assumed to be stationary white noise. The equation of motion can be expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{v}}(t) + \boldsymbol{C}\dot{\boldsymbol{v}}(t) + \boldsymbol{K}\boldsymbol{v}(t) = \boldsymbol{l}w(t) \tag{18}$$

where $\boldsymbol{v}(t)$, $\dot{\boldsymbol{v}}(t)$, and $\ddot{\boldsymbol{v}}(t)$ are the stationary displacement, velocity, and acceleration responses, respectively, and $\boldsymbol{l}$ is a vector for which the elements are the influence factors for each DOF and may be thought of a measure of the extent to which the $w(t)$ participates in the total excitation on the structure. Multiplying both sides of Eq. (18) by a slowly-time-varying amplitude-modulating function $\Gamma(t)$, we can obtain

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{C}\dot{\boldsymbol{u}}(t) + \boldsymbol{K}\boldsymbol{u}(t) = \boldsymbol{l}f(t) \tag{19}$$

where $f(t)$ is a nonstationary white noise as represented by the product model,

$$f(t) = \Gamma(t)w(t) \tag{20}$$

and $\boldsymbol{u}(t) = \Gamma(t)\boldsymbol{v}(t)$. Note that in deriving Eq. (19), we assumed that $\Gamma(t)$ is a slowly-time-varying function [i.e., $\dot{\Gamma}(t) \approx 0$, $\ddot{\Gamma}(t) \approx 0$], and so $\Gamma(t)\dot{\boldsymbol{v}}(t) \approx \dot{\boldsymbol{u}}(t)$ and $\Gamma(t)\ddot{\boldsymbol{v}}(t) \approx \ddot{\boldsymbol{u}}(t)$. The results indicate that if the excitation can be modeled as nonstationary white noise as represented in Eq. (20) with a slowly-time-varying envelope function $\Gamma(t)$, then the nonstationary responses of the system can also be modeled approximately as a product model with the same envelope function.

To transform the original nonstationary responses into stationary ones and to circumvent the practical problem of evaluating nonstationary correlation from very limited data samples, we extract the amplitude-modulating function from the original nonstationary data, which can be done by evaluating the temporal root-mean-square functions from the real data. The theoretical background is given as follows [15].

Denote the time average of $u_i^2(\tau)$ as $\hat{u}_i^2(t)$, which is defined a

$$\hat{u}_i^2(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} u_i^2(\tau)\,\mathrm{d}\tau \tag{21}$$

Recall that we have assumed the $\Gamma(\tau)$ to be a slowly varying function, then from Eq. (21), $\hat{u}_i^2(t)$ can be approximated as

$$\hat{u}_i^2(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \Gamma^2(\tau)v_i^2(\tau)\,\mathrm{d}\tau \cong \Gamma^2(t)\frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} v_i^2(\tau)\,\mathrm{d}\tau \tag{22}$$

for T being a short time interval. The temporal mean-square function $\hat{u}_i^2(t)$ is practically estimated by averaging over short time intervals of the record. If we assume that $v_i(\tau)$ is an ergodic process, the integral on the right-hand side of Eq. (22) is just an approximation to $E[v_i^2]$ and so

$$\hat{u}_i^2(t) \cong \Gamma^2(t)E[v_i^2] \tag{23}$$

Then the temporal root-mean-square function denoted as $\hat{\Psi}_i(t)$ can be evaluated by time-averaging over a single sample record as

$$\hat{\Psi}_i(t) = [\hat{u}_i^2(t)]^{\frac{1}{2}} \cong \Gamma(t)C_i \tag{24}$$

where $C_i = (E[v_i^2])^{\frac{1}{2}}$. Note that the temporal root-mean-square function $\hat{\Psi}_i(t)$ of each DOF is proportional to the same envelope function of time $\Gamma(t)$.

The preceding result indicates that the temporal root-mean-square functions $\hat{\Psi}_i(t)$ of the response histories describe the same time variation as given by the envelope function $\Gamma(t)$. This suggests that if the original nonstationary data could be represented by the product model with a slowly varying envelope function, the temporal root-mean-square functions of the data also have the same nonstationary trend as that of the original data. The temporal root-mean-square function $\hat{\Psi}_i(t)$, and so the envelope function $\Gamma(t)$, can thus be determined by using the interval average and then applying curve-fitting. We can then acquire the approximate stationary responses by dividing the nonstationary responses of each DOF with the same envelope function $\Gamma(t)$. The correlation functions of the stationary response data can then be obtained, which are in turn treated as the free-decay responses corresponding to each DOF. The modal parameters of the system can then be obtained via a time-domain modal-identification method, such as the ITD method, as described next.

Ibrahim Time-Domain Modal-Identification Method

The ITD method uses free-decay responses of a structure under test to identify its modal parameters in complex form [9]. From the measured free responses at n stations on a structure under test, each with q sampling points, we define a system matrix $\boldsymbol{S}$, which is an $n \times n$ matrix, such that

$$\boldsymbol{S}\boldsymbol{X} = \boldsymbol{Y} \tag{25}$$

where $\boldsymbol{X}$ is an $n \times q$ matrix of measured response, and $\boldsymbol{Y}$ is an $n \times q$ matrix of time-delayed response. Generally, the number q is chosen to be larger than the number of measurement channels n. Therefore, the system matrix $\boldsymbol{S}$ can be estimated by the least-squares method.

In theory, a continuum structure has an infinite number of degrees of freedom and an infinite number of modes. In practice, we do not know in advance how many modes are required to describe the dynamic behavior of the observed structural system. However, the important modes of the system under consideration could be roughly found by examining the Fourier spectra associated with the measured response histories. The number of (real) modes m involved in the response then determines the order n of the system matrix $\boldsymbol{S}$ in Eq. (25), for which n is chosen to be at least twice of the number of modes of interest to appropriately identify the $2m$ complex modes. Note that n is not necessarily the same as the number of measured DOF of the system. If the number of measurement channels does not actually reach n, we may employ the technique of channel expansion [9], which uses time-delayed sampling points from the original response as new response channels, so that the total number of measurement channels can reach n. Note, however, that the identified mode shapes are composed of the components corresponding only to those physically measured response channels.

It can be shown that the natural frequencies and the damping ratios of the original vibrating system are directly related to the eigenvalues of the system matrix $\boldsymbol{S}$, and the mode shapes correspond to the eigenvectors of $\boldsymbol{S}$. Denote an eigenvalue of $\boldsymbol{S}$ and a characteristic root of the original vibrating system as $\rho_r = \beta_r + i\gamma_r$ and $s_r = \sigma_r + i\upsilon_r$, respectively. One can derive [9]

$$\begin{cases} \sigma_r = \frac{1}{2\Delta t}\ln(\beta_r^2 + \gamma_r^2) \\ \upsilon_r = \frac{1}{\Delta t}\tan^{-1}\left(\frac{\gamma_r}{\beta_r}\right) \end{cases} \tag{26}$$

from which the natural frequencies and damping ratios of the structural system can be obtained to be

$$\begin{cases} \omega_{nr} = \sqrt{\sigma_r^2 + \upsilon_r^2} \\ \zeta_r = \frac{|\sigma_r|}{\sqrt{\sigma_r^2 + \upsilon_r^2}} \end{cases} \tag{27}$$

Hence, once the system matrix $\boldsymbol{S}$ is obtained via least-squares analysis from measured data, the modal parameters of the structural system can be determined by solving the eigenvalue problem associated with the system matrix $\boldsymbol{S}$.

Numerical Simulation

To demonstrate the effectiveness of the proposed method, we first consider a linear 6-DOF chain model with viscous damping. A schematic representation of this model is shown in Fig. 1. The mass matrix $\boldsymbol{M}$, stiffness matrix $\boldsymbol{K}$, and damping matrix $\boldsymbol{C}$ of the system are given as follows:

$$\boldsymbol{M} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \mathrm{N \cdot s^2/m},$$

$$\boldsymbol{K} = 600 \cdot \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & -2 & 5 \end{bmatrix} \mathrm{N/m},$$

$$\boldsymbol{C} = 0.05\boldsymbol{M} + 0.001\boldsymbol{K} + 0.2 \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}_{6\times 6} \mathrm{N \cdot s/m}$$

Note that the system has nonproportional damping (and so complex modes in general), because the damping matrix $\boldsymbol{C}$ cannot be expressed as a linear combination of $\boldsymbol{M}$ and $\boldsymbol{K}$. Consider that the ambient vibration input can be modeled as nonstationary white noise, as represented by the product model given by Eq. (13). The stationary white noise is generated using the spectrum approximation method [16] as a zero-mean bandpass noise, for which the power spectral density constant is $0.02\ \mathrm{m^2/(s^3 \cdot rad)}$ with a frequency range from 0 to 50 Hz. The sampling interval is chosen as $\Delta t = 0.01$ s, and the sampling period is $T = 1310.72$ s. The stationary white noise simulated is then multiplied by an amplitude-modulating function $\Gamma(t) = 4 \cdot (e^{-0.002t} - e^{-0.004t})$ to obtain the nonstationary white noise, which serves as the excitation input acting on the sixth mass point of the system. The time signal of a simulated sample of the nonstationary white noise and the power spectrum of the corresponding stationary part are shown in Figs. 2 and 3, respectively.

The displacement responses of the system were obtained using Newmark's method. By examining the Fourier spectrum associated with each of the response channels, we chose the response of the sixth channel $X_6(t)$, which contains rich overall frequency information, as the reference response to compute the correlation functions of the system. According to the theory presented in the previous sections, the nonstationary problem may reduce to a stationary problem if we can extract the amplitude-modulating function from the original nonstationary data. Therefore, we can follow the same procedures as those for stationary problems, and the correlation functions thus obtained are treated as free-vibration data. The Ibrahim time-domain method could then be applied to identify modal parameters of the system.

The results of modal-parameter identification are summarized in Table 1, which shows that the errors in natural frequencies are less than 1% and the errors in damping ratios are less than 5%. Note that the "exact" modal damping ratios listed in Table 1, as well as the exact mode shapes, are actually the equivalent values obtained by using ITD from the simulated free-vibration data of the nonproportionally damped structure. The identified mode shapes are also compared with the exact values in Fig. 4, in which we observe good agreement, and the maximum error is about 15%. The errors of identified damping ratios and mode shapes are somewhat higher, which may be due to the fact that the system response generally has lower sensitivity to these modal parameters than to the modal frequencies. It should be mentioned that the selection of reference channel for computing correlation functions is important to the identification results. The richer the frequency content of the

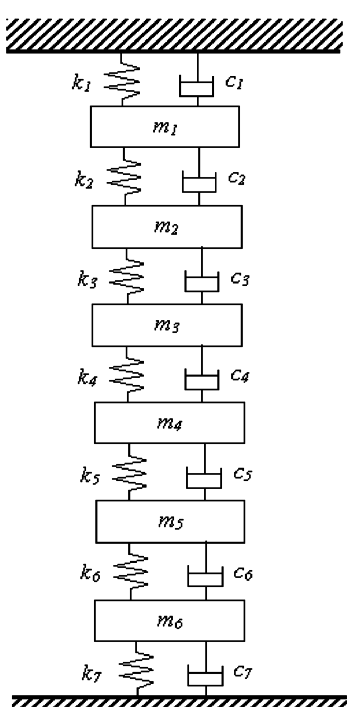

Fig. 1 Schematic plot of the 6-DOF chain system.

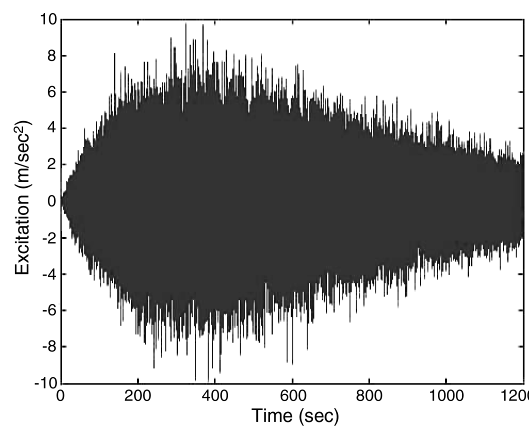

Fig. 2 Sample function of nonstationary white noise with a slowly varying amplitude-modulating function.

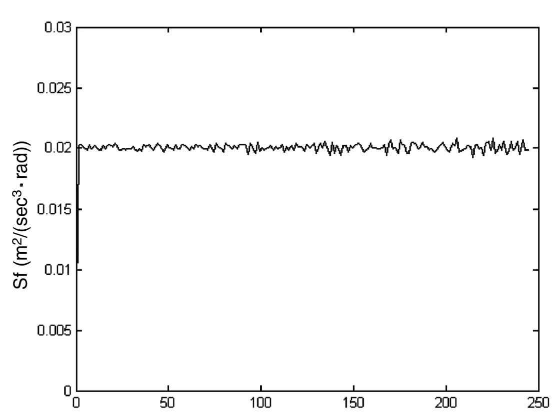

Fig. 3 Power spectrum associated with the stationary part of the simulated nonstationary white noise.

Table 1 Results of modal-parameter identification of a 6-DOF system subjected to nonstationary white-noise input

	Natural frequency, rad/s			Damping ratio, %			
Mode	Exact	ITD	Error %	Exact	ITD	Error, %	MAC
1	5.03	5.03	0.06	5.24	5.12	2.29	1.00
2	13.45	13.43	0.16	1.07	1.06	0.93	1.00
3	19.80	19.74	0.28	1.13	1.09	3.54	1.00
4	26.69	26.53	0.60	1.43	1.41	1.40	0.98
5	31.66	31.41	0.80	1.66	1.65	0.60	0.94
6	33.73	33.38	1.03	1.74	1.70	2.30	0.95

reference channel, the better the modal-parameter identification that can be achieved.

In the preceding, we considered the nonstationary excitation to be nonstationary white noise modeled as the product of stationary white noise and an amplitude-modulating function. This restriction could be removed by treating the nonstationary excitation as nonstationary color noise or filtered white noise. Nonstationary color noise is modeled as the product of an amplitude-modulating function and a stationary color noise, which is in turn obtained as the output of a certain system (acting as a filter) to an input of stationary white noise. A simulated sample of nonstationary color noise is shown in Fig. 5, in which the filter system was assumed to be a second-order system with a frequency of 21.691 rad/s and a damping ratio of 5%. Results of modal-parameter identification of the 6-DOF system subjected to nonstationary color input are summarized in Table 2, in which the fourth mode is the mode of the filter system of excitation and therefore has no corresponding exact mode shape for computing the modal assurance criterion (MAC) value [cf. Eq. (28)]. Note in this case that the modes identified by ITD generally include the vibrating modes of the structural system, the excitation modes of the filter system, and some fictitious modes due to numerical computation. From identification results, as shown in Table 2, it is seen that the system characteristics as well as the input characteristics were both identified. We can distinguish the vibrating modes of the structural system from the excitation modes and the fictitious modes if the mass

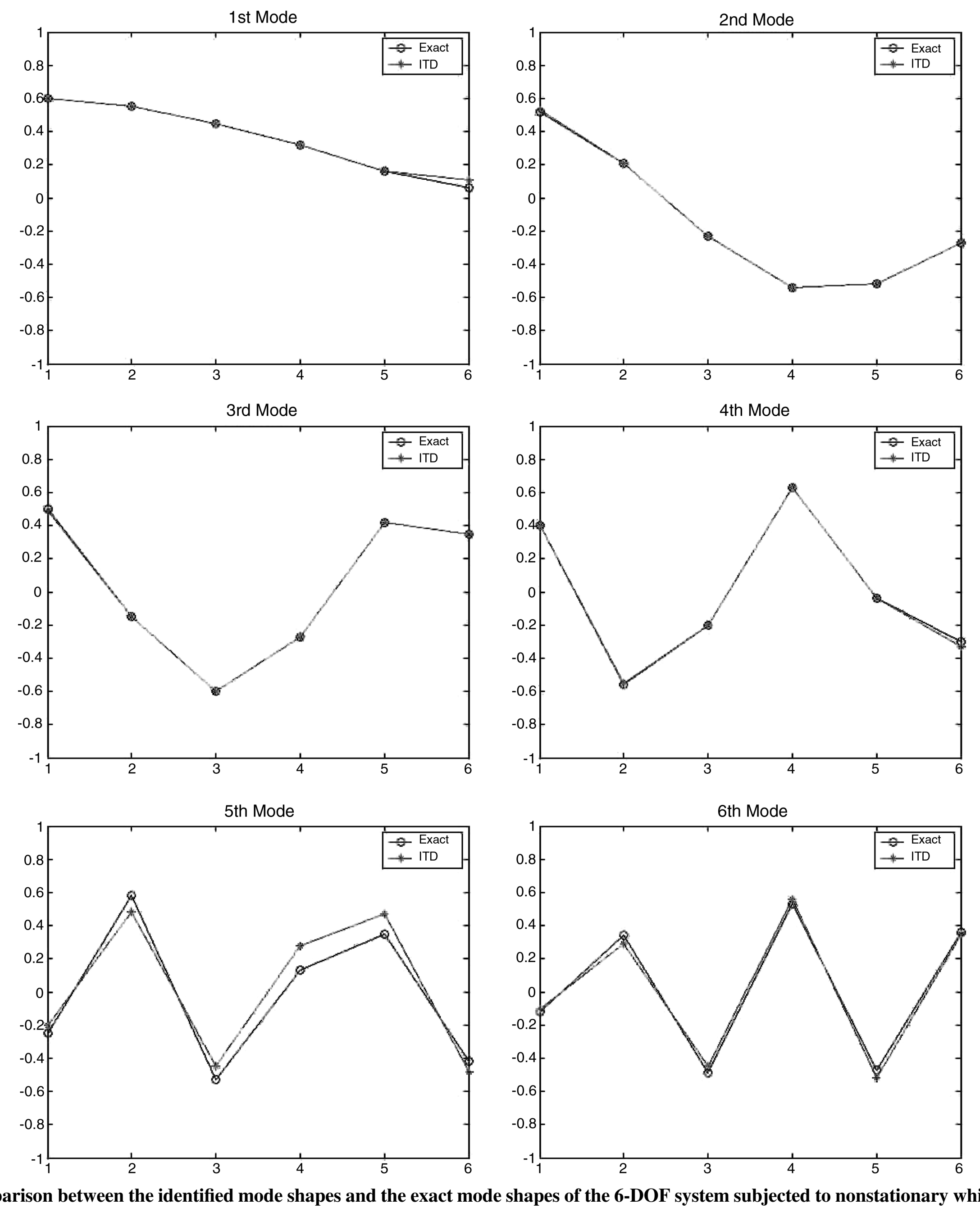

Fig. 4 Comparison between the identified mode shapes and the exact mode shapes of the 6-DOF system subjected to nonstationary white-noise input.

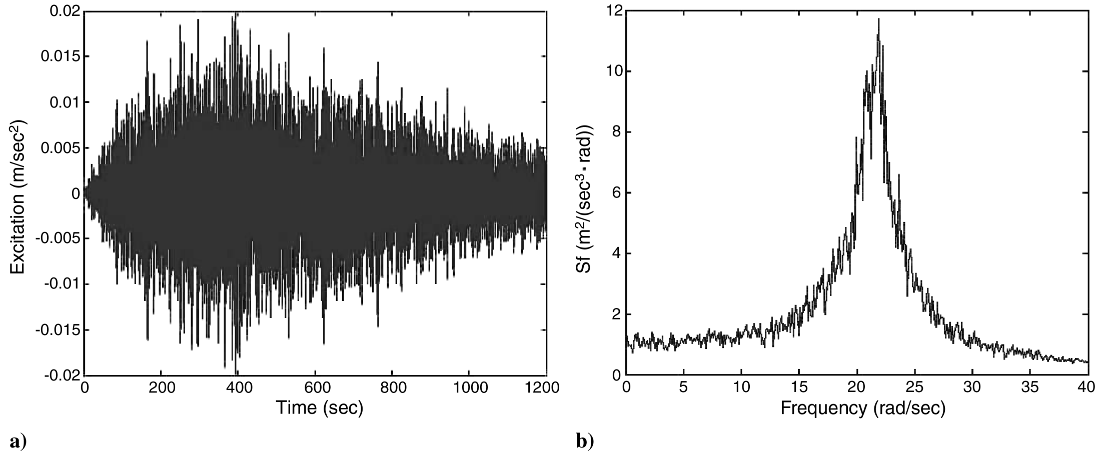

Fig. 5 Sample of nonstationary input of color noise: a) time history and b) power spectrum (stationary part).

matrix or the stiffness matrix of the structural system is available according to the orthogonality conditions, which show that vibrating shapes are orthogonal with respect to the stiffness matrix as well as to the mass matrix.

Furthermore, to check for agreement between the identified mode shapes and exact shapes, we use the MAC [17] that has been extensively used in the experimental modal analysis. The definition of the MAC is

$$\mathrm{MAC}\,(\boldsymbol{\Phi}_{iA}, \boldsymbol{\Phi}_{jX}) = \frac{|\{\boldsymbol{\Phi}_{iA}\}^T\{\boldsymbol{\Phi}_{jX}\}^*|^2}{\{\boldsymbol{\Phi}_{iA}\}^T\{\boldsymbol{\Phi}_{iA}\}^*\{\boldsymbol{\Phi}_{jX}\}^T\{\boldsymbol{\Phi}_{jX}\}^*} \tag{28}$$

where $\boldsymbol{\Phi}_{iA}$ and $\boldsymbol{\Phi}_{jX}$ represent two mode-shape vectors of interest, and the superscript * denotes the complex conjugate. The value of MAC varies between 0 and 1. When the MAC value is equal to 1, the two vectors $\boldsymbol{\Phi}_{iA}$ and $\boldsymbol{\Phi}_{jX}$ represent exactly the same mode shape. On the other hand, when two mode shapes are orthogonal with each other, the MAC value is zero.

In the previous examples, we considered lightly damped structural systems. To address the role of system damping in the numerical accuracy of modal-parameter identification, numerical simulation of a 6-DOF chain model with heavy viscous damping has been performed. The mass matrix $\boldsymbol{M}$ and the stiffness matrix $\boldsymbol{K}$ of the system are as indicated in the previous example. The system damping matrix is assumed:

$$\boldsymbol{C} = 0.8\boldsymbol{M} + 0.008\boldsymbol{K} + 0.2\begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}_{6\times 6} \quad \mathrm{N \cdot s/m}$$

The system considered has heavy damping ratios (around 8–15% of critical damping), as listed in Table 3, in which the results of identification are also shown. It can be observed that, in general, the accuracy in frequency identification is acceptable, but the errors in damping ratios get large, especially for the higher modes. This

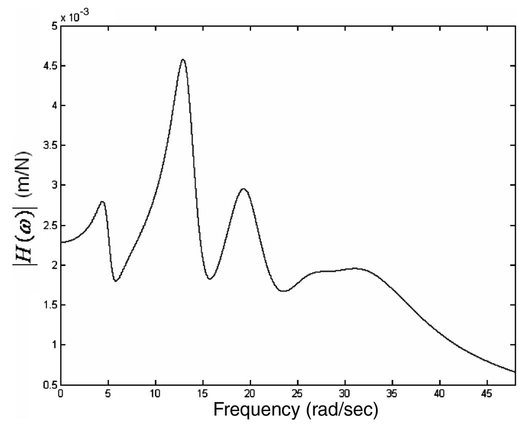

Fig. 6 Typical plot of the amplitude frequency response function $[H_{66}(\omega)]$ of the system showing modal interference among higher modes.

situation becomes more obvious when the system damping gets even heavier. A typical plot of the amplitude frequency response function $[H_{66}(\omega)$, which denotes the frequency response of the displacement at the sixth DOF due to the input at the sixth DOF] of the system is shown in Fig. 6, in which we clearly see the serious problem of modal interference, especially among the higher modes. To summarize, if a system has relatively heavy damping, it becomes difficult to accurately identify the damping ratios and the mode shapes, due to the interference among modes of the system. The proposed method can yield reasonably accurate results when modal damping ratios of the system are below 10%.

Recall that in this paper, we introduced a slowly-time-varying amplitude-modulating function $\Gamma(t)$ to model the nonstationary data. The solution procedure proposed involves the extraction of the modulating function from the response data, after which the modal parameters can then be identified. This could lead to a problem of accuracy if the dynamic system of interest has a mode of low-frequency, comparable with the major frequency of the modulating

Table 2 Results of modal-parameter identification of a 6-DOF system subjected to nonstationary color input

	Natural frequency, rad/s			Damping ratio, %			
Mode	Exact	ITD	Error %	Exact	ITD	Error, %	MAC
1	5.03	5.03	0.06	5.24	5.07	3.24	0.94
2	13.45	13.42	0.22	1.07	0.99	7.48	1.00
3	19.80	19.74	0.29	1.13	1.15	1.77	1.00
4	21.69	21.82	0.58	5.00	2.86	42.80	——[a]
5	26.69	26.51	0.67	1.43	1.38	3.50	1.00
6	31.66	31.42	0.76	1.66	1.60	3.61	0.97
7	33.73	33.34	1.14	1.74	1.63	6.32	0.43

[a]The fourth mode is a fictitious mode of excitation and has no corresponding exact mode shape for comparison.

Table 3 Results of modal-parameter identification of a 6-DOF system containing heavy damping ratios subjected to nonstationary white-noise input

	Natural frequency, rad/s			Damping ratio, %			
Mode	Exact	ITD	Error %	Exact	ITD	Error, %	MAC
1	5.03	5.02	0.19	14.44	15.21	5.38	1.00
2	13.45	13.24	1.58	8.54	8.77	2.78	0.87
3	19.80	19.09	3.57	9.89	9.33	5.69	0.80
4	26.69	25.06	6.08	12.07	8.18	32.22	0.54
5	31.66	29.14	7.94	13.46	21.06	56.54	0.13
6	33.73	31.30	7.19	14.39	6.03	58.10	0.01

function itself. To address this issue, numerical simulation has been performed of a linear 6-DOF chain model subjected to nonstationary white-noise input. From the Fourier spectrum associated with $\Gamma(t)$, the modulating function of the excitation, the statistical average frequency (or expected frequency) [18] of $\Gamma(t)$ is evaluated to be 1.36×10^{-2} rad/s, which is considered as the major frequency of $\Gamma(t)$. The mass matrix $\boldsymbol{M}$, stiffness matrix $\boldsymbol{K}$, and the damping matrix $\boldsymbol{C}$ of the system are given as follows:

$$\boldsymbol{M} = 10,000 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} \text{N} \cdot \text{s}^2/\text{m},$$

$$\boldsymbol{K} = 200 \cdot \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 0 & -4 & 7 \end{bmatrix} \text{N/m},$$

$$\boldsymbol{C} = 0.04\boldsymbol{M} + 0.006\boldsymbol{K} + 0.02 \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}_{6\times 6} \text{N} \cdot \text{s/m}$$

The results of modal identification are summarized in Table 4, in which we note that the lowest modal frequency of the system considered is about 0.01 Hz and the identification for modal damping is much poorer than that for frequency. The situation is very different when a system without modes of very low frequencies is considered (cf. Table 1). This difference in performance of the proposed method could be attributed to the following fact. Once the modulating function $\Gamma(t)$ was extracted from the response data with error, this error would lead to a distortion in the vibration amplitudes of the stationary parts of the system response. This amplitude distortion would then lead to error in damping identification, which is especially significant for the modes with very low frequencies.

To further examine the effectiveness of the present method for a more complex structural system, we conduct modal-parameter identification analysis using a 20-DOF chain model with nonproportional damping. Assume that each mass is 1 kg and all spring constants are 600 N/m. The damping matrix of the system is assumed to be

$$\boldsymbol{C} = 0.05\boldsymbol{M} + 0.001\boldsymbol{K} + 0.02 \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}_{20\times 20} \text{N} \cdot \text{s/m}$$

The system considered has many groups of closely spaced modes, as listed in Table 5. The results of identification are also summarized in Table 5, which shows that the errors in natural frequencies are less than 5% and the errors in damping are larger. Observing the MAC values, which signify the consistency between the identified and the theoretical mode shapes, we found that 11 out of the 20 modes are identified accurately (MAC $\geq$ 0.9). The errors of identified damping ratios and mode shapes are somewhat larger, due to the fact that the system response generally has lower sensitivity to these modal parameters than to the modal frequencies. It is also noted that, in general, the higher modes are not identified as accurately as the lower modes, because their contribution to the system response is somewhat less than that of the lower modes.

Table 4 Results of modal-parameter identification of a 6-DOF system containing low-frequency modes subjected to nonstationary white-noise input

	Natural frequency, rad/s			Damping ratio, %			
Mode	Exact	ITD	Error %	Exact	ITD	Error, %	MAC
1	0.06	0.06	0.06	3.37	18.28	441.78	1.00
2	0.22	0.23	0.23	0.91	0.37	58.92	1.00
3	0.37	0.37	0.37	0.55	1.26	127.73	1.00
4	0.49	0.47	0.47	0.42	3.21	658.95	0.90
5	0.59	0.59	0.59	0.36	0.74	107.49	1.00
6	1.25	1.26	1.26	0.20	0.15	24.76	1.00

Table 5 Results of modal-parameter identification of a 20-DOF system subjected to nonstationary white-noise input

	Natural frequency, rad/s			Damping ratio, %			
Mode	Exact	ITD	Error %	Exact	ITD	Error, %	MAC
1	3.66	3.65	0.27	5.49	5.25	4.34	1.00
2	7.30	7.30	0.02	0.71	0.67	5.45	1.00
3	10.90	10.88	0.18	0.94	0.98	3.91	1.00
4	14.44	14.41	0.19	0.89	0.76	14.55	1.00
5	17.90	17.86	0.22	1.06	0.99	7.19	1.00
6	21.26	21.20	0.28	1.17	1.24	5.53	1.00
7	24.49	24.37	0.50	1.33	1.34	1.49	0.98
8	27.60	27.38	0.79	1.45	1.87	29.08	0.96
9	30.54	30.38	0.55	1.59	1.95	22.50	0.95
10	33.32	32.94	1.14	1.71	1.76	2.82	0.95
11	35.91	35.27	1.80	1.83	1.59	13.10	0.91
12	38.30	37.84	1.22	1.94	2.15	11.06	0.77
13	40.48	39.84	1.57	2.03	1.55	23.63	0.67
14	42.43	42.05	0.88	2.12	1.67	21.15	0.55
15	44.14	42.44	3.85	2.20	1.64	25.28	0.05
16	45.60	45.02	1.28	2.26	2.52	11.72	0.18
17	46.81	46.46	0.76	2.26	0.47	79.09	0.01
18	47.76	47.24	1.09	2.20	1.83	16.95	0.07
19	48.44	47.12	2.73	2.51	0.21	91.63	0.05
20	48.85	48.78	0.15	2.41	0.19	92.12	0.09

Conclusions

To identify dynamic characteristics of structures in nonstationary ambient vibration, modal analysis of using only measured responses is studied. It is shown that if the input signals can be modeled as nonstationary white noise, which is a product of white noise and a deterministic time-varying function, the theoretical nonstationary correlation functions evaluated at a fixed time instant of structural response will have the same mathematical form as free vibration of the structure. The practical problem of insufficient data samples available for evaluating nonstationary correlation can be approximately resolved by first extracting the amplitude-modulating function from the response and then transforming the nonstationary responses into stationary ones. The correlation functions of the stationary response are treated as free-vibration response, and so the Ibrahim time-domain method can then be applied to identify modal parameters of the system. In addition, the choice of the reference channel for computing the correlation functions is important to the identification results. The reference channel is chosen as a response channel for which the Fourier spectrum has rich frequency content around the structure modes of interest. The richer the frequency content of the reference channel, the better the modal-parameter identification that can be achieved. Modal-parameter identification using ambient data excited by nonstationary color noise is also considered. This is accomplished via adding, in cascade, a pseudoforce system to the structural system under consideration. Identification results are then sorted as either structural parameters or input-force characteristics using the orthogonality conditions of structure modes. To verify the validity of the present identification procedure, numerical simulations have been performed for various cases in which the systems considered may have heavy damping, low frequency, or more degrees of freedom, respectively. It was demonstrated that, in general, the proposed method can yield reasonably accurate results of modal-parameter identification. However, if the system has relatively heavy damping or has very low natural frequencies comparable with the major frequency of the amplitude-modulating function of the excitation, it could become difficult to identify some of the modal parameters using the proposed method.

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